

# Roughness and multiscaling of planar crack fronts

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**Abstract.** We consider numerically the roughness of a planar crack front within the long-range elastic string model, with a tunable disorder correlation length  $\xi$ . The problem is shown to have two important length scales,  $\xi$  and the Larkin length  $L_c$ . Multiscaling of the crack front is observed for scales below  $\xi$ , provided that the disorder is strong enough. The asymptotic scaling with a roughness exponent  $\zeta \approx 0.39$  is recovered for scales larger than both  $\xi$  and  $L_c$ . If  $L_c > \xi$ , these regimes are separated by a third regime characterized by the Larkin exponent  $\zeta_L \approx 0.5$ . We discuss the experimental implications of our results.

## 1. Introduction

Crack fronts propagating along a weak plane of a disordered material provide an ideal experimental system for studying phenomena such as the depinning transition, the associated avalanche dynamics and roughness of the crack front. Experimentally, by studying systems such as sintered Plexiglas plates [1, 2, 3] and paper sheets [4], it has been established that a slowly driven planar crack front propagates in a sequence of avalanches, with a power law size distribution. In the case of the Plexiglas experiments the crack fronts have been observed to be rough [1, 2].

As is often the case when studying similar phenomena typically exhibiting some degree of universality, it is expected that the statistical properties of the planar crack fronts can be theoretically described with a simplified model, which here is given by the long-range elastic string [5, 6]. Such a model has successfully reproduced the statistics of avalanches of crack front propagation [7], and in particular the local clusters such avalanches are broken into due to the long range elastic interactions between different segments of the crack front [8]. Early experiments [1, 2] found a roughness exponent significantly larger than that predicted by the crack line model [6, 9, 10], but a more recent study has shown that the theoretically predicted value of the roughness exponent is recovered if large enough length scales are considered [11]. Moreover, the short length scale regime has been found to exhibit multiscaling [11].

In this paper we consider the long range elastic string driven in a random potential with a tunable disorder correlation length, in order to clarify the origin and nature of the different scaling regimes of the crack front roughness. The main benefit of our model as compared to previous studies of the crack line model is that the disorder correlation length  $\xi$  can be chosen to be larger than the crack line segment length, thus making it possible to study also the regime below  $\xi$ . It is demonstrated that the problem includes two relevant length scales, the disorder correlation length  $\xi$  and the Larkin length  $L_c$  [12]. By considering different sets of parameters of the model, the relative magnitudes of these two length scales can be tuned. For strong enough disorder (or equivalently soft enough lines), multiscaling of the crack front is observed for scales below  $\xi$ . Asymptotically, for length scales exceeding both  $\xi$  and  $L_c$ , we recover the well known roughness exponent  $\zeta \approx 0.39$  of the long-range elastic string [6, 9, 10]. An intermediate regime characterized by the Larkin exponent  $\zeta_L \approx 0.5$  is observed above  $\xi$  and below  $L_c$ , if  $\xi < L_c$ . The paper is organized as follows: In the next Section, the details of the numerical model are presented. In Section 3 we present a theoretical description of the expected scaling regimes of the front roughness, while Section 4 includes detailed numerical results from simulations of the crack line model. Section 5 finishes the paper with conclusions and discussion.

## 2. Model

The model of a propagating planar crack front considered here is represented by a vector of single-valued integer heights  $h_i$ ,  $i = 1, \dots, L$ , with  $L$  the system size [6]. Crack propagation is driven by the local stress intensity factor (SIF)  $K_i$ , representing the asymptotic prefactor of the  $1/\sqrt{r}$  divergence if the stress field near the crack tip.  $K_i$  is taken to be of the form  $K_i = K_i^{\text{elastic}} + K_{i,h_i}^{\text{random}} + K^{\text{ext}}$ . Here

$$K_i^{\text{elastic}} = \Gamma_0 \sum_{j \neq i}^L \frac{h_j - h_i}{b|j - i|^2} \quad (1)$$

is the first-order variation of the stress intensity factor due to first-order perturbation of the front position [5], with  $b$  the front segment spacing and  $\Gamma_0$  tunes the strength of the long-range elastic interactions. For periodic boundary conditions along the crack front as considered here, this becomes [13, 14]

$$K_i^{\text{elastic}} = \Gamma_0 b \left( \frac{\pi}{L} \right)^2 \sum_{j \neq i}^L \frac{h_j - h_i}{\sin^2(|j - i|b\pi/L)}. \quad (2)$$

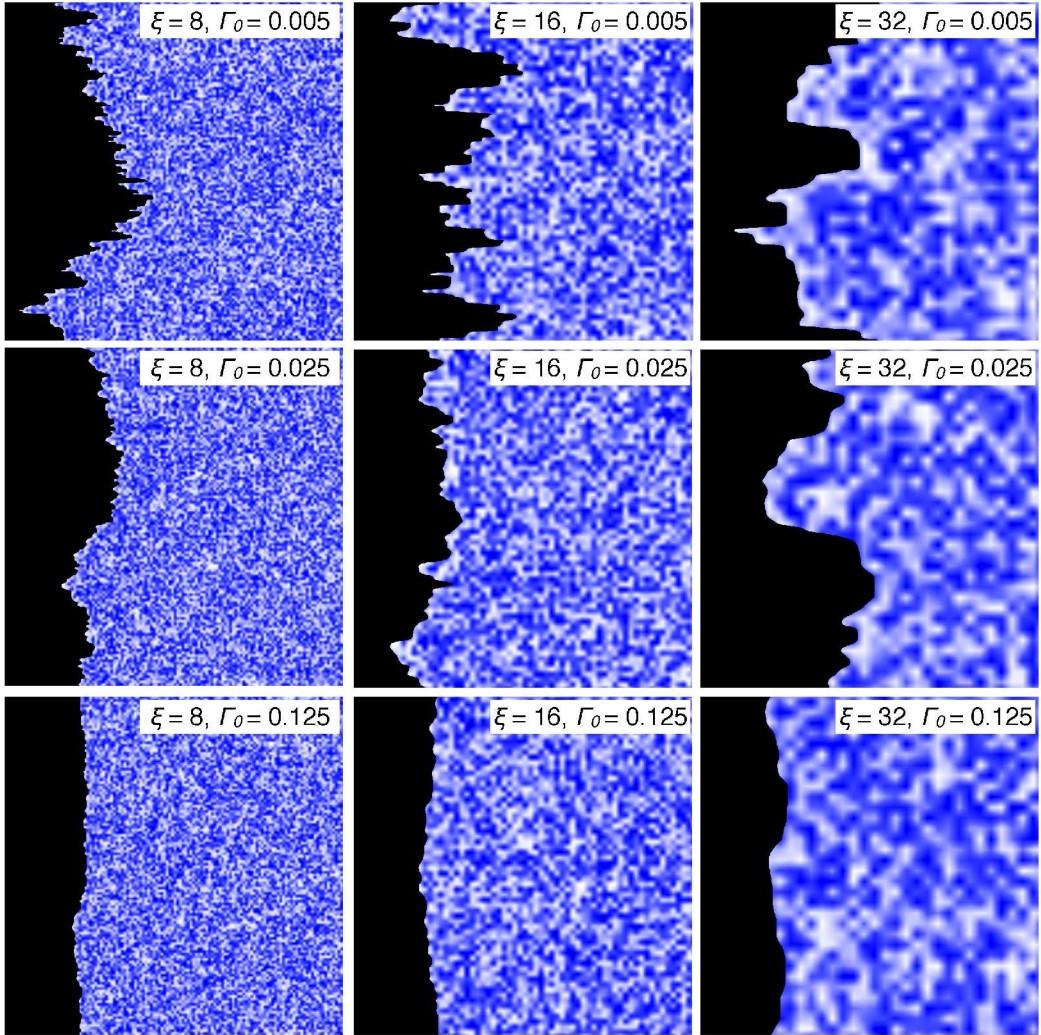
$K_{i,h_i}^{\text{random}}$  is a time-independent disorder field with correlations

$$\langle K_{i,h_i}^{\text{random}} K_{j,h_j}^{\text{random}} \rangle - \langle K_{i,h_i}^{\text{random}} \rangle \langle K_{j,h_j}^{\text{random}} \rangle \sim f(r/\xi), \quad (3)$$

where  $r = \sqrt{(i-j)^2 + (h_i - h_j)^2}$  and  $f(x) = 1$  for  $x \ll 1$  and  $f(x) = 0$  for  $x \gg 1$ . In practice, such a disorder field can be prepared by assigning uncorrelated random numbers from a distribution to a subset of lattice sites forming a square grid with a spacing of  $\xi$ , and filling the rest of the lattice sites by using an appropriate interpolation algorithm. Here, we use a uniform distribution from -1 to 0, and bilinear interpolation to obtain a smooth disorder field, see Fig. 1 for examples.  $K^{\text{ext}}$  is the contribution of the external load. The dynamics is defined in discrete time by setting

$$v_i(t) = h_i(t+1) - h_i(t) = \theta(K_i), \quad (4)$$

where  $v_i$  is the local velocity of the front element  $i$ , and  $\theta$  is the Heaviside step function. Parallel dynamics is assumed, such that during a single time step, all the front elements with  $v_i > 0$  are advanced by a unit step,  $h_i(t+1) = h_i(t) + 1$ . The local forces are then recomputed for each element, and the process is repeated until  $v_i = 0$  for all  $i$  and the avalanche stops. The applied load is then increased so that exactly one front element becomes unstable (i.e.  $v_i > 0$ ), and a new avalanche is initiated. As the crack front advances, the applied SIF  $K^{\text{ext}}$  decreases at a rate proportional to the instantaneous average velocity  $v(t) = 1/L \sum_{i=1}^L v_i(t)$  of the front, with a proportionality constant  $\epsilon$ . Such a protocol leads to a cut-off for the avalanche size distribution scaling with  $\epsilon$  [8], and corresponds to quasistatic driving, which has the advantage that observables such as the avalanche sizes can be defined without the need to apply (a possibly ambiguous) non-zero threshold. Notice that while the discretized dynamics we employ here neglects the fact that the local velocity of the crack front should be proportional to the local SIF, this simplification is well known to have no influence on the scaling behaviour we study here.



**Figure 1.** Snapshots of crack fronts from the line model moving from left to right, with various values of  $\xi$  and  $\Gamma_0$ . For each case, a square area of the weak plane of linear size  $L = 1024$  is shown. The cracked region is displayed in black. The different shades of blue indicate different values of the local toughness of the non-cracked material: dark (light) blue areas correspond to weak (strong) material.

### 3. Roughness of the crack front due to disorder

As the planar crack front advances along the weak plane, it will roughen due to disorder. Such disorder, originating from the fluctuations of the local toughness of the weak plane, is characterized by a fluctuation amplitude  $R$  and a correlation length  $\xi$ . The crack front morphology can be characterized by considering the structure functions [11, 15]

$$C_k(\delta) = \langle |h_{i+\delta} - h_i|^k \rangle_i^{1/k}. \quad (5)$$

If the front fluctuations follow Gaussian statistics, the structure functions with different  $k$  can be collapsed by normalizing the  $C_k$ 's with the Gaussian factors [11, 15]

$$R_k^G = \sqrt{2} \left[ \frac{\Gamma((k+1)/2)}{\sqrt{\pi}} \right]^{1/k}. \quad (6)$$

Close to the depinning transition (i.e. when a slow enough external drive is applied), we expect the roughness of the front to be controlled by two length scales,  $\xi$  and the Larkin length  $L_c \sim \Gamma_0^2 \xi / R^2$  [8, 16]. In the following, we will consider the different regimes separately.

### 3.1. Strong pinning: multiscaling of the front

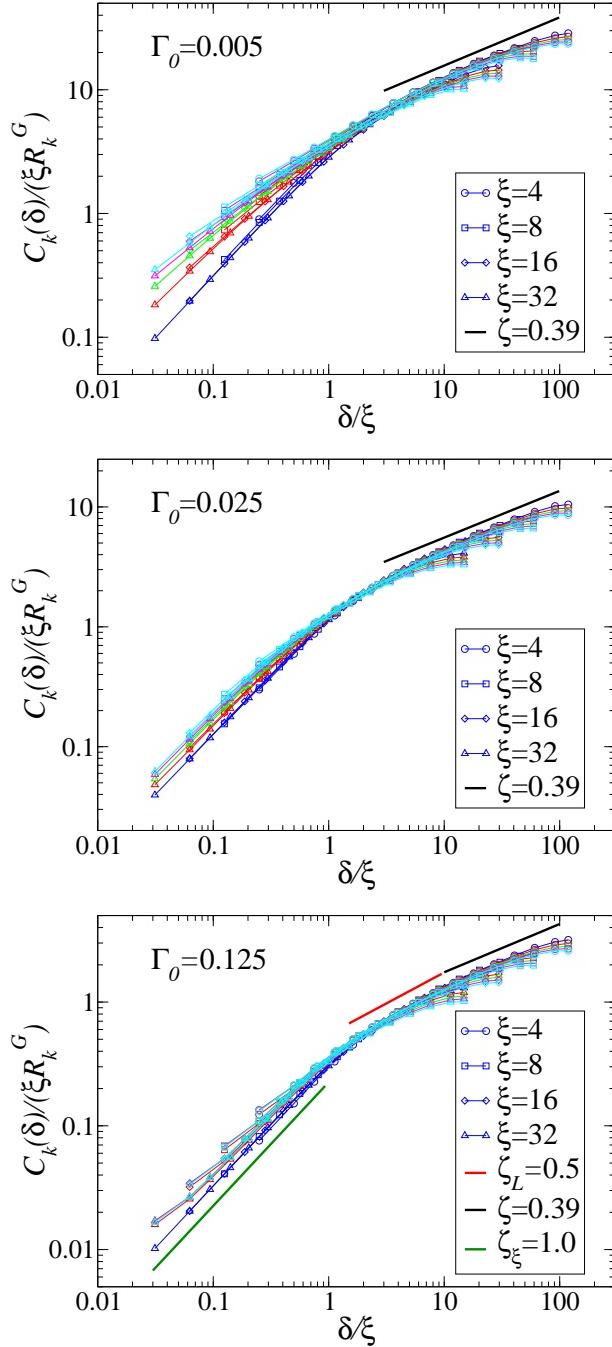
For small enough  $\Gamma_0$  (or equivalently, large enough  $R$ ), the disorder is strong enough to substantially deform the front locally, which may lead to steep slopes (and possibly even overhangs in experiments and in models which would allow them to be formed) of the crack front, and thus to multiscaling of the front roughness, or  $k$ -dependent scaling of the structure functions  $C_k(\delta)$ . In the spirit of the idea of strong pinning (individual “pinning centers”, or correlated fluctuations of the random potential, are strong enough to substantially deform the front), we expect such large deformations and steep slopes of the crack front to take place up to a lateral length scale proportional to the disorder scale  $\xi$ . This regime corresponds to a small Larkin length as compared to the scale of the toughness fluctuations,  $L_c \ll \xi$ . Thus, we expect to observe multiscaling for scales below  $\xi$ , and the asymptotic scaling characterized by the unique roughness exponent  $\zeta \approx 0.39$  for scales larger than  $\xi$ . By tuning  $\Gamma_0$  within this regime, the amplitude of the local deformations and thus the strength of the multiscaling can be changed.

### 3.2. Weak pinning: the Larkin regime

In the opposite limit, where  $L_c \gg \xi$ , the toughness fluctuations are weak and no significant deformation of the crack front takes place for lateral scales of the order of  $\xi$ . Instead, the front remains essentially undeformed for lateral scales below  $L_c$ , and the potential energy fluctuations follow Poissonian statistics. This leads to front roughness characterized by the Larkin exponent  $\zeta_L = 1/2$  for scales above  $\xi$  and below  $L_c$ . For scales above  $L_c$  the toughness fluctuations become effective and the asymptotic roughness exponent  $\zeta \approx 0.39$  is expected.

## 4. Numerical results

We simulate the crack line model in a system of linear size  $L = 1024$ , by considering different disorder correlation lengths ( $\xi = 4, 8, 16$  and  $32$ ) and values of the crack front stiffness ( $\Gamma_0 = 0.005, 0.025$  and  $0.125$ ). The value  $\epsilon = 0.001$  is used for the parameter controlling the avalanche cut-off scale. Fig. 1 shows examples of the crack front profiles for different  $\xi$  and  $\Gamma_0$ . Also the toughness fluctuations are shown, with dark (light) blue corresponding to regions with low (high) local toughness.



**Figure 2.** The structure functions  $C_k(\delta)$  rescaled according to Eq. (7). Various values of  $\xi$  (different symbols, as indicated by the legends) and  $k$  (different colors; blue, red, green, magenta and cyan correspond to  $k = 1, 2, 3, 4$  and  $5$ , respectively) are considered. Data for three values of the stiffness parameter  $\Gamma_0$  are shown:  $\Gamma_0 = 0.005$  (top),  $\Gamma_0 = 0.025$  (middle), and  $\Gamma_0 = 0.125$  (bottom).

In the spirit of the discussion of the previous Section, we consider the scaling form

$$C_k(\delta) = R_k^G \xi F(\delta/\xi) \quad (7)$$

to account for the effect of varying  $\xi$  for fixed  $\Gamma_0$ . Here,  $F(x)$  is expected to scale as

$$F(x) \sim x^\zeta \quad (8)$$

for  $x > 1$  and  $L_c \ll \xi$ , and

$$F(x) \sim x^{\zeta_L} \quad \text{for } 1 < x < L_c/\xi, \quad (9)$$

$$F(x) \sim x^\zeta \quad \text{for } x > L_c/\xi \quad (10)$$

for  $L_c \gg \xi$ . Fig. 2 shows the rescaled  $C_k(\delta)$  functions for three different values of  $\Gamma_0$ . For  $\delta/\xi > 1$ , the data collapse nicely, verifying the scaling form, Eq. (7), and the Gaussian nature of the large scale front roughness. Also the different scaling regimes as given by Eqs. (8), (9) and (10) are clearly visible, with  $\zeta \approx 0.39$  and  $\zeta_L \approx 0.5$ . Notice also that no multiscaling is observed, as long as only scales above  $\xi$  are considered.

For  $\delta/\xi < 1$  and small  $\Gamma_0$ , the data displays multiscaling, a signature of a deviation from the Gaussian statistics. The strength of this multiscaling depends on  $\Gamma_0$ . This indicates that the multiscaling regime may not be universal: different parameters of the model could correspond to different effective exponents within this regime. For large enough  $\Gamma_0$ , the  $\delta/\xi < 1$  regime is characterized by a linear dependence of  $C_k(\delta)$  on  $\delta$ , or  $C_k(\delta) \sim \delta^{\zeta_\xi}$  with  $\zeta_\xi \approx 1.0$ , and essentially no multiscaling is observed: The small deviation from linear behaviour for the smallest values of  $\delta$  in the bottom panel of Fig. 2 is due to the discrete nature of the model.

## 5. Conclusions and discussion

In this paper we have studied the roughness of a planar crack front driven in a disordered medium, by considering the standard crack line model, with the additional ingredient of using toughness fluctuations with a characteristic scale  $\xi$  which may be tuned. Such a simple approach has made it possible to identify the relevant length scales of the problem, and to propose an explanation of the experimental results regarding the crack front roughness.

Based on our results, we think that the experiments reported in Ref. [11] correspond to the "strong pinning" regime, where the toughness fluctuations cause substantial local deformations of the crack front, visible as multiscaling for short length scales. We expect the cross-over scale separating this multiscaling regime from the asymptotic scaling regime characterized by the roughness exponent  $\zeta \approx 0.39$  to be proportional to the disorder scale  $\xi$ . Indeed, in the experiments reported in Ref. [11], the cross-over scale is shifted towards larger values when the diameter of the beads used for the sand blasting process to prepare the experimental samples is increased. Moreover, the spatial resolution of the experiment is clearly high enough to observe features of the crack front also below scales corresponding to the diameter of the beads [11]. However, it is not clear how the statistical properties of the toughness fluctuations depend on the preparation procedure of the samples, making it difficult to make a precise comparison between theory and experiment.

Future directions of research related to the present topic would include to develop an experimental setup in which the toughness fluctuations could be controlled, and thus the effect of their statistical properties could be tested properly. Regarding the theoretical side, it could be useful to develop a model in which a more accurate expression for the long range interaction kernel would be used, accurately describing interactions between segments of a crack front of an arbitrary shape [17]. Such a more realistic line model for the crack front could also include the possibility to form overhangs. However, we think that the picture regarding the different scaling regimes and the cross-over scales separating them presented in this paper would hold also in that case.

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